Fastest Climb of a Piston-Prop Aircraft

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Nomenclature

A, A^*	=	$(b/a)^{1/3}$ and $(b\sigma_{cr}/a)^{1/3}$
a^*	=	a/σ_{cr}
a, b	=	notations given by the relations of Eq. (11)
C_D, C_L	=	drag and lift coefficients of aircraft, $C_D = C_{D0} + KC_L^2$
C_{D0}	=	zero-lift drag coefficient
$\hat{c}$	=	power-specific fuel consumption
E	=	C_L/C_D, aerodynamic efficiency of aircraft
h	=	altitude
K	=	induced drag factor
k	=	conversion factor
P, P_e	=	thrust power and engine power, $P = k\eta_p P_e$
R/C	=	rate of climb
S	=	wing planform area
t	=	endurance, time taken in climb
V	=	constant true airspeed during climb
W	=	total weight of aircraft
x	=	range, horizontal distance traveled in climb
β	=	9296 m in troposphere, and β = 7254 m in stratosphere
γ	=	climb angle
ζ	=	$(W_1 - W_2)/W_1$, climb-fuel weight fraction
η_p	=	propeller efficiency
ρ	=	density of atmospheric air
σ	=	ρ/ρ_{SSL}, density ratio

Subscripts

cr	=	critical altitude, at about 6 km
FC	=	fastest climb
m	=	maximum
Pmin	=	minimum power
Pmin, SSL	=	minimum power at SSL
SSL	=	standard atmosphere at sea level
1, 2	=	beginning of climb and end of climb

Introduction

THE maximum rate of climb is a figure of merit of aircraft. Equations relating airspeed, climb angle, and rate of climb have been recently published by Hale.[1] He proceeds further to obtain range and endurance by introducing approximations in the functional forms of γ_{FC} and $(R/C)_{FC}$ of the aircraft. This Note does not use these approximations, but improves the analysis of fastest climb and obtains analytical expressions for the horizontal distance covered, time taken, and the fuel consumed during the fastest climb. This is directly useful in performance calculations for the preliminary design of a piston-prop aircraft. Finally, the analysis is applied to a given aircraft and its fastest climb performance is presented graphically.

Basic Relations of Climb Performance

Consider a piston-prop aircraft climbing from h_1 to another higher h_2 in steady straight flight. The climb parameters γ, R/C, x, t, and ζ are obtained[1] as

$$\gamma = \frac{P}{WV} - \frac{1}{E} = k\eta_p(P_e/W)\frac{1}{V} - \frac{1}{E} \tag{1}$$

$$R/C = \frac{P}{W} - \frac{V}{E} = k\eta_p(P_e/W) - \frac{V}{E} \tag{2}$$

$$x = \int_{h_1}^{h_2} \frac{dh}{\gamma}, \quad t = \int_{h_1}^{h_2} \frac{dh}{R/C}, \qquad h_2 \geq h_1 \tag{3}$$

$$\xi = 1 - \exp\left\{-\hat{c}\int_{h_1}^{h_2} \frac{P_e/W}{R/C}\,dh\right\}, \qquad h_2 \geq h_1 \tag{4}$$

where $\hat{c}$ is considered constant and γ is assumed small as is generally the case in practice. The value of k depends on the system of units used. If P_e is expressed in hp, the value of k = 550 ft lb/s hp. In the system of units if P_e is expressed in watts then the value of k = 1; if P_e is given in kilowatts then k = 1000 W/kW.

The variation of P_e with altitude (and thus σ) depends on whether the engine is aspirated or supercharged (turbocharged). This variation is usually approximated such that it gives quick results for the preliminary design of an aircraft. It is, therefore, common to use the following reasonable approximations. In the case of aspirated engines

$$P_e/P_{e,\mathrm{SSL}} = \sigma \tag{5}$$

and in the case of supercharged engines

$$P_e/P_{e,\mathrm{SSL}} = 1, \text{ below the critical altitude} \tag{6}$$

$$P_e/P_{e,\mathrm{SSL}} = \sigma/\sigma_{cr}, \text{ above the critical altitude} \tag{7}$$

The variation of σ with altitude is provided[1] for the standard atmosphere as

$$\sigma = e^{-h/\beta} \tag{8}$$

where β = 9296 m (30,500 ft) in the troposphere and β = 7254 m (23,800 ft) in the stratosphere.

Airspeed, Climb Angle and Rate of Climb

These flight parameters are obtained by Hale,[1] and therefore, their deductions are briefly presented here in the form which is used in the subsequent sections. The fastest climb airspeed V_{FC} is obtained by maximizing R/C of Eq. (2) with respect to V. This gives

$$V_{FC} = \left[\frac{2(W_1/S)}{\rho_{SSL}\sigma}\right]^{1/2}\left(\frac{K}{3C_{D0}}\right)^{1/4} = V_{P\min} = \frac{V_{P\min,\mathrm{SSL}}}{\sigma^{1/2}} \tag{9}$$

$$E_{FC} = E_{P\min} = 0.866E_m \tag{10}$$

For the sake of brevity the notations a and b can be introduced as

$$a = k\eta_p(P_{e,\mathrm{SSL}}/W), \quad \text{and} \quad b = V_{P\min,\mathrm{SSL}}/0.866E_m \tag{11}$$

where both a and b are constant quantities for a given aircraft. Introducing Eqs. (9) and (10) in Eq. (1) and using the notations of Eq. (11), γ_{FC} can be expressed as

$$\gamma_{FC} = \frac{1}{V_{P\min,\mathrm{SSL}}}\left(\frac{P_e}{P_{e,\mathrm{SSL}}}a\sigma^{1/2} - b\right) \tag{12}$$

In the case of aspirated engines, the above relation becomes

$$\gamma_{FC} = \frac{1}{V_{P\min,\mathrm{SSL}}}(a\sigma^{3/2} - b) \tag{13}$$

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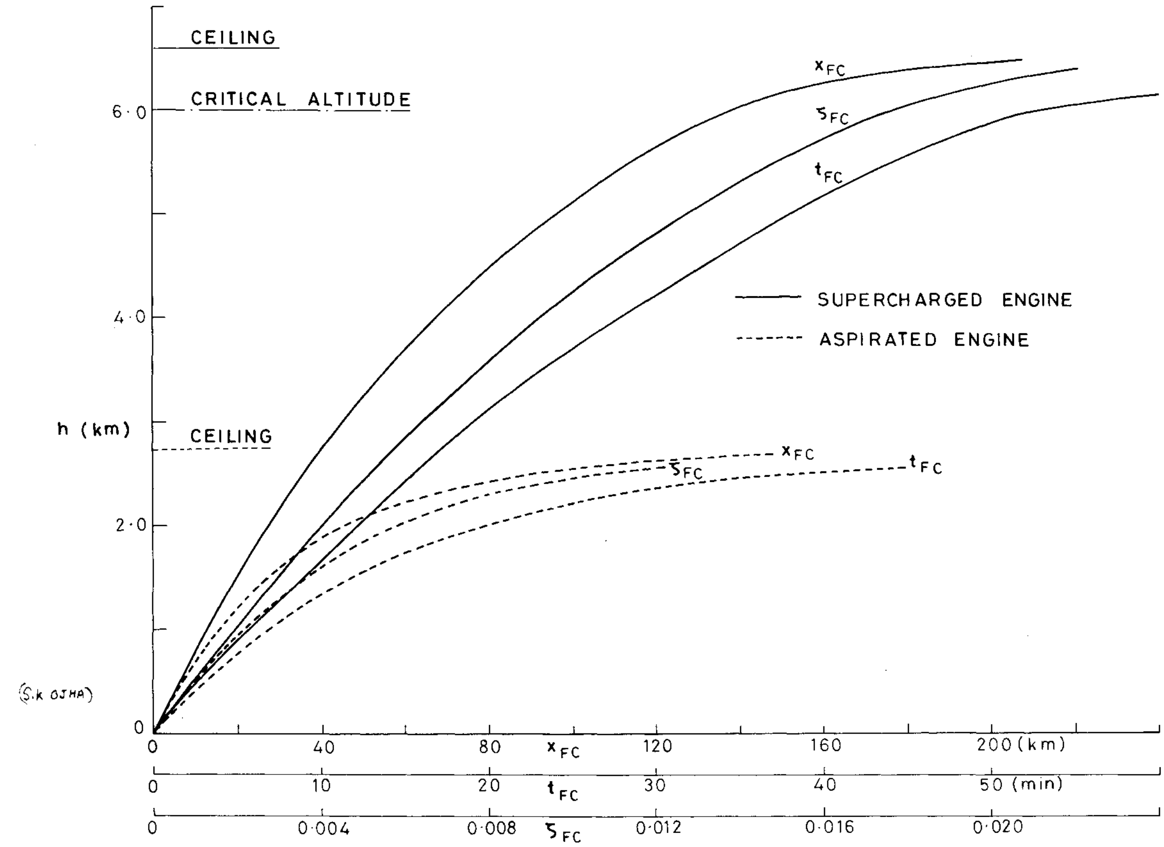

Fig. 1 Variations of x_{FC}, t_{FC}, and ζ_{FC} with altitude for aspirated and supercharged engines.

and similarly in the case of supercharged engines Eq. (12) becomes

$$\gamma_{FC} = \frac{1}{V_{P\min,SSL}}(a\sigma^{1/2} - b), \quad \text{for} \quad h \le h_{cr} \tag{14}$$

$$\gamma_{FC} = \frac{1}{V_{P\min,SSL}}\left(\frac{a}{\sigma_{cr}}\sigma^{3/2} - b\right), \quad \text{for} \quad h > h_{cr} \tag{15}$$

The $(R/C)_m$ is obtained by introducing Eqs. (9) and (10) in Eq. (2) and using the notations of Eq. (11). This gives

$$(R/C)_m = (R/C)_{FC} = \left(\frac{P_e}{P_{e,SSL}}\right) a - \frac{b}{\sigma^{1/2}} \tag{16}$$

In the case of aspirated engines the above relation becomes

$$(R/C)_m = a\sigma - b/\sigma^{1/2} \tag{17}$$

and in the case of supercharged engines Eq. (16) becomes

$$(R/C)_m = a - b/\sigma^{1/2}, \quad \text{for} \quad h \le h_{cr} \tag{18}$$

$$(R/C)_m = a(\sigma/\sigma_{cr}) - b/\sigma^{1/2}, \quad \text{for} \quad h > h_{cr} \tag{19}$$

where σ_{cr} is supposed to be a known quantity and σ is obtained either from Eq. (8) or from the standard atmospheric table.

Range

The range of climbing flight is the horizontal distance covered during the climb. The range x_{FC} of the fastest climb is obtained from the first relation of Eq. (3) as

$$x_{FC} = \int_{h_1}^{h_2} \frac{dh}{\gamma_{FC}}, \quad \text{where} \quad h_2 \ge h_1 \tag{20}$$

For an aspirated engine, the use of Eqs. (13) and (8) will allow the integration of the right side of the above relation to obtain

$$x_{FC} = V_{P\min,SSL}\frac{2\beta}{3b}\ln\frac{be^{3h_1/2\beta} - a}{be^{3h_2/2\beta} - a} \tag{21}$$

Similarly, for a supercharged engine

$$x_{FC} = V_{P\min,SSL}\frac{2\beta}{b}\ln\frac{be^{h_1/2\beta} - a}{be^{h_2/2\beta} - a}, \quad \text{for} \quad h_2 \le h_{cr} \tag{22}$$

$$x_{FC} = V_{P\min,SSL}\frac{2\beta}{3b}\ln\frac{be^{3h_1/2\beta} - a/\sigma_{cr}}{be^{3h_2/2\beta} - a/\sigma_{cr}}, \quad \text{for} \quad h_1 \ge h_{cr} \tag{23}$$

Endurance

The endurance t_{FC} of the fastest climb is obtained from the second relation of Eq. (3) as

$$t_{FC} = \int_{h_1}^{h_2} \frac{dh}{(R/C)_m}, \quad \text{where} \quad h_2 \ge h_1 \tag{24}$$

For an aspirated engine, the use of Eqs. (17) and (8) will permit the integration of the right side of the above relation in closed form. Therefore, the t_{FC} can be expressed as

$$t_{FC} = \frac{2\beta A}{3b}\left[\sqrt{3}\left(\tan^{-1}\frac{2e^{-h_2/2\beta} + A}{A\sqrt{3}} - \tan^{-1}\frac{2e^{-h_1/2\beta} + A}{A\sqrt{3}}\right) + \frac{1}{2}\ln\frac{e^{-h_2/\beta} + Ae^{-h_2/2\beta} + A^2}{e^{-h_1/\beta} + Ae^{-h_1/2\beta} + A^2} - \ln\frac{e^{-h_2/2\beta} - A}{e^{-h_1/2\beta} - A}\right] \tag{25}$$

Similarly, for a supercharged engine

$$t_{FC} = \frac{2\beta}{a} \ln \frac{ae^{-h_1/2\beta} - b}{ae^{-h_2/2\beta} - b}, \quad \text{for} \quad h_2 \le h_{cr} \tag{26}$$

$$t_{FC} = \frac{2\beta A^*}{3b}\left[\sqrt{3}\left(\tan^{-1}\frac{2e^{-h_2/2\beta} + A^*}{A^*\sqrt{3}} - \tan^{-1}\frac{2e^{-h_1/2\beta} + A^*}{A^*\sqrt{3}}\right) + \frac{1}{2}\ln\frac{e^{-h_2/\beta} + A^*e^{-h_2/2\beta} + A^{*2}}{e^{-h_1/\beta} + A^*e^{-h_1/2\beta} + A^{*2}} - \ln\frac{e^{-h_2/2\beta} - A^*}{e^{-h_1/2\beta} - A^*}\right], \quad \text{for} \quad h_1 \ge h_{cr} \tag{27}$$

Fuel Consumption

The amount of fuel consumed during climb from h_1 to h_2 is expressed in nondimensional form ζ which is called the climb-fuel weight fraction. Its value ζ_{FC} for the fastest climb is obtained from Eq. (4) as

$$\zeta_{FC} = 1 - \exp\left[-\hat{c}\int_{h_1}^{h_2} \frac{P_e/W}{(R/C)_m}\,\mathrm{d}h\right], \qquad h_2 \ge h_1 \tag{28}$$

For aspirated engines, the use of Eqs. (5), (17), and (8) allows the integration of the right side of the above equation in analytical form. Therefore, the above relation becomes

$$\zeta_{FC} = 1 - \exp\left[\frac{2\beta\hat{c}(P_{e,SSL}/W)}{3a} \ln \frac{ae^{-3h_2/2\beta} - b}{ae^{-3h_1/2\beta} - b}\right] \tag{29}$$

Similarly, for supercharged engines

$$\zeta_{FC} = 1 - \exp\left[\frac{2\beta\hat{c}(P_{e,SSL}/W)}{a} \ln \frac{ae^{-h_2/2\beta} - b}{ae^{-h_1/2\beta} - b}\right]$$

$$\text{for} \quad h_2 \le h_{cr} \tag{30}$$

$$\zeta_{FC} = 1 - \exp\left[\frac{2\beta\hat{c}(P_{e,SSL}/W)}{3a} \ln \frac{ae^{-3h_2/2\beta} - b\sigma_{cr}}{ae^{-3h_1/2\beta} - b\sigma_{cr}}\right]$$

$$\text{for} \quad h_1 \ge h_{cr} \tag{31}$$

The notations a and b are obtained from Eq. (11), and σ_{cr} is also supposed to be a known quantity before proceeding for calculations of the climb performance of a piston-prop aircraft.

Application to an Aircraft

A piston-prop aircraft of total weight of 7350 N (1652.3 lb), wing area of 11.2 m^2 (120.5 ft^2) and engine power of 75 kW (100.6 hp) has the parabolic drag polar $C_D = 0.032 + 0.14C_L^2$. Its power-specific fuel consumption is 0.55×10^{-6} N/s W (0.33 lb/h hp), propeller efficiency of 0.85 and a critical altitude at 6 km. Find the range, endurance, and fuel weight fraction as the aircraft passes through different known altitudes during the fastest climb in standard atmosphere, both for aspirated and supercharged engines.

The above aircraft has wing loading (W/S) of 656.25 N/m^2, engine power-to-weight ratio ($P_{e,SSL}/W$) of 10.204 W/N, and maximum aerodynamic efficiency E_m of 7.47. Since the flight is confined in the troposhere, $\beta = 9296$ m. From the standard atmospheric table, $\rho_{SSL} = 1.226$ kg/m^3 and $\sigma_{cr} = \rho_{cr}/\rho_{SSL} = 0.538$ at the critical altitude of 6000 m. The use of Eq. (9) gives $V_{Pmin,SSL} = 35.96$ m/s and from the two equations of relation (11), $a = 8.673$ and $b = 5.558$.

It is assumed that the steady straight climbing flight has started from the ground at sea level itself, therefore, $h_1 = 0$ in Eqs. (21), (22), (25), (26), (29), and (30). The h_2 would be regarded as the known variable h ($\ge h_1$). Using Eqs. (21), (25), and (29), respectively, x_{FC}, t_{FC}, and ζ_{FC} are obtained at different altitudes for aspirated piston-prop engine aircraft and these are shown in Fig. 1 as the three lower curves. In the case of supercharged (turbocharged) engines, the x_{FC}, t_{FC}, and ζ_{FC} are obtained from Eqs. (22), (26), and (30) for below the critical altitude, and from Eqs. (23), (27), and (31) for above the critical altitude. These results are also shown in Fig. 1 and they form the three upper curves of the figure. The curves for both aspirated and supercharged engines flatten out as they approach their ceiling altitudes because of the decrease in rate of climb.

Reference

[1]Hale, F. J., *Aircraft Performance, Selection, and Design*, 1st ed., Wiley, New York, 1984, pp. 141–151.

Special Rotation Vectors: A Means for Transmitting Quaternions in Three Components

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Introduction

THE use of quaternions[1–3] for computing and recording the orientation of a body in three dimensions[4,5] has been gaining acceptance. This method has clear advantages over traditional Euler angles.[6–8] The differential equations governing the time development of quaternions are free of singularities for all orientations; trigonometric functions are eliminated from the differential equations as well as from the process of constructing a rotation matrix. In terms of computational load, these advantages clearly outweigh the need to treat four, rather than three, components and to enforce an auxiliary condition against truncation errors.

There is one situation, however, where the trade between quaternions and Euler angles is not clearly decided. This is the case when orientation information is passed over a data link. Here, quaternions require an additional one-third of bandwidth. In some cases bandwidth is a scarcer commodity than computational throughput, and this might tip the balance in favor of Euler angles.

In this Note we introduce a scheme for eating your cake and having it too. We show how it is possible to capture quaternion style orientation information in a three component special rotation vector (SRV). The mathematical theory of SRVs and their relationship to the group of rotations in three dimensions will be covered elsewhere.[9] The purpose of this Note is to provide the practical rules for using SRVs for transmitting quaternion-style orientation information. The method of using quaternions in this context is outlined in Appendix A. Between the body of the text and Appendix A this Note is self-contained. It can be understood and applied without reading Ref. 9. The correspondence between SRVs and rotations is specified in Appendix B.

Quaternions consist of a vector part $\boldsymbol{Q}$ that has three components and a scalar part Q_0. The four components are used in Eq. (A14) to produce the rotation matrix. They are propagated in time by the differential Eqs. (A18) and (A19).

Now suppose that two computers, both using quaternions to represent orientation, need to pass this information to each

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